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ON THE MOTION OF A BALL ON A BILLIARD TABLE.

By N. J. LENNES, University of Chicago.

A ball, of radius r, is assumed to move in straight lines and without friction on a rectangular billiard-table, the lengths of whose sides are a and b. Under the assumption that the angle of reflection is equal to the angle of incidence, it is required to discuss completely the motion of the center of the ball when started from a given point and in a given direction.

Under the conditions of the problem the center of the ball must always remain at least at a distance r from the sides of the table, *i. e.*, the center cannot move outside a rectangle whose sides are $a-2r$ and $b-2r$. The problem is obviously equivalent to the problem in which a ball of zero radius is considered to move on a table of dimensions $a-2r$ and $b-2r$. Denote $a-2r$ and $b-2r$ by m and n, and denote the corners of this last rectangle in their cyclic order by A, B, C, D. When we say that the ball strikes one of the sides of the table, we mean that the center of the ball is on one of the segments AB, BC, CD, DA. When the ball strikes a corner of the table the center is at A, B, C, or D. The angle at which the ball strikes the side of the table is the angle between the line of motion and the perpendicular to that side.

The following propositions are immediate consequences of the conditions of the problem:

(1). If the ball is reflected back and forth several times between two opposite sides without striking the remaining sides then the points at which it strikes one side are equidistant and all the lines of motion from either one of these sides to the other are parallel.

(2). If the ball is reflected from one side, as BC, to an adjacent side, as CD, the line of reflection from the side CD is parallel to the line of incidence with BC.

(3). If the ball strikes a corner it is reflected back along the line on which it approached the corner.

(4). If the ball is reflected from a side, as BC, to an adjacent side CD, then it strikes the side CD the same distance from the point C that it would if the table were so extended that the ball would move freely in the direction in which it moved before it struck BC until it should meet the side CD extended.

It follows directly from (3) that if the ball strikes a corner it will retrace its path. These considerations enable us to investigate the periodicity of the motion by considering the problem in the following simplified form, viz., we consider a table in which one side is removed and the table extended indefinitely in that direction. Let the side removed be one whose length is m. On the two extended sides lay off the distance n an indefinite number of times beginning at the corners of the original table. These points are referred to as multiple points of n. The motion of the ball on the original table may now be regarded as transformed into the motion of the ball on the extended table. It is quite easy to see, from (1)—(4), that if the ball strikes one of the multiple points of n on the extended table then in the corresponding motion on the original table it would strike a corner, and conversely if the ball strikes a corner on the original table then it strikes a multiple point of n on the extended table.

Suppose first that the ball is started from the corner A in a direction whose angle with AB is α. Let the side CD opposite AB be removed and the table extended indefinitely in that direction. Under what conditions will the ball strike a multiple point of n? The answer is: If $m\tan\alpha$ and n are commensurable, the length of AB being m and that of BC being n. If $m\tan\alpha$ and n have a common measure l, which is contained h times in $m\tan\alpha$ and k times in n, then $km\tan\alpha=hn$. Consequently the ball will strike a multiple point of n at a distance hn from the side AB. We may assume that l is the greatest common measure of $m\tan\alpha$ and n. Hence h and k are integers prime to each other. It follows that hn will be the first multiple point of n which the ball strikes. If k is an even number this multiple point of n will be on the side AD extended (the side from which the ball was started).

In every case when the ball strikes the first multiple point of n on the side AD extended, h is odd; for h and k being prime to each other cannot both be even. By direct consideration of the phenomenon on the table this appears from the fact that if h were even we should have by the symmetry of the motion that the ball would strike an odd multiple of n on the side BC half way between the side AB and the multiple point of n in question. This corresponds to the fact that on the original table the ball cannot return to the point A without returning along the path on which it left A; that it cannot strike A without first striking another corner. If k is odd the first multiple point of n reached is on the side BC extended. If h and k are both odd, then the ball will strike the side BC extended in an odd multiple point of n, which means that in the original motion it would strike the corner C. If k is odd and h even the ball will strike the side BC in an even multiple point of n, which corresponds to the point n on the original table.

If k is even the ball will strike each of the sides AD and BC $\frac{1}{2}k$ times in a half period (counting the starting point A but not the point D). If k is odd, it strikes each of the sides AD and BC $\frac{1}{2}(k+1)$ times (in this case counting both the starting point and the point of final impact). Similarly, if h is even the ball will strike each of the sides AB and CD $\frac{1}{2}h$ times counting the starting point but not the point of final impact, and if h is odd it strikes each of the sides AB and CD $\frac{1}{2}(h+1)$ times (counting both the point of departure and the point of final impact).

If k is even and h is odd the first half period ends at D, if h is even and k is odd it ends at B, and if both h and k are odd it ends at C.

If the ball is started from a point P, on the side AD, P being different from A, there are four possibilities, viz., (a) $m\tan\alpha$ is commensurable with n but not with AP or PD. (b) $m\tan\alpha$ is commensurable with n and with either AP or PD (in this case it follows as a theorem that $m\tan\alpha$ is commensurable with both AP and PD). (c) $m\tan\alpha$ incommensurable with n but commensurable with one of the segments AP and PD (in this case it is a theorem that $m\tan\alpha$ is incommensurable with one of the segments AP and PD). (d) $m\tan\alpha$ incommensurable with n and with both of the segments AP and PD. We consider each case separately.

(a) If $m\tan\alpha$ is commensurable with n but not with AP or PD, then obviously the ball cannot strike a multiple point of n and hence not a corner on the original table but will strike points whose distance from a multiple point of n is AP. If h and k are integers such that $km\tan\alpha=hn$, and if k is even the ball will strike the side AD at a point whose distance from A is $AP+hn$. (We denote by P' all points whose distance from A is $AP+In$ where I is any integer). This corresponds to striking the point P on the original table. Since the ball cannot reach P along the line on which it was started from P but must reach it in a direction whose angle with the side AD is equal to that at which it left P it follows that the ball completes a period on reaching P.

If k is odd the ball will strike a point P' on the side AD which again corresponds to striking P on the closed table.

There is no half period. The number of impacts against each of the sides AD and BC is $\frac{1}{2}k$ if k is even (counting the point P once), and if k is odd (again counting the P once). Similarly, it strikes each of the sides AB and CD $\frac{1}{2}h$ times if k is even, and h times if k is odd.

(b) Under (b) two cases are possible, viz., (1) The ball may strike a point P' on the side AD before it strikes a multiple point of n. The result is the same as in (a). (2) If the ball strikes a multiple point of n before it strikes a point P' it comes under the case discussed above where the ball is started from a corner.

(c) In case $m\tan\alpha$ if incommensurable with n but commensurable with one of the segments AP and PD, then obviously the ball cannot reach a point P'. Further, it can reach at most one corner on the original table; consequently the motion is not periodic in this case.

(d) In case $m\tan\alpha$ is incommensurable with n and both segments AP and PD the motion is obviously not periodic.

In case the motion is not periodic every point on the table lies in the path of the ball or is a limit point of such points. This can be proved by means of the following lemma: *Given any two irrational numbers a and b and any positive number e, however small, then it is possible* to find a pair of integers A and B such that* $| Aa+Bb | < e$.

Consider now the case where the ball started from A. According to the lemma for any preassigned positive number e there exists a pair of integers h and k such that $| km\tan\alpha-hn | < e$. This shows that on the extended table the ball comes as near as we please to multiple points of n. If k is even, D is obviously a limit point of the set of points passed over by the ball. In the same manner we can show that any point on either one of the sides AD or BC

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*Jules Tannery, *Theorie des Fonctions*, 2d ed., I, p. 37.

which is not reached by the ball is a limit point of points which it does reach. Since the paths of the ball from one side to the opposite side are parallel it follows that every point on the table not reached by the ball is a limit point of points which it does reach.

Evidently the same conclusion follows in the non-periodic cases in which the ball starts from a point other than a corner of the table.

DEPARTMENTS.

NOTE. All solutions of problems, problems for solution, and other department contributions should be sent direct to THE AMERICAN MATHEMATICAL MONTHLY, 1227 Clay Street, Springfield, Mo.

SOLUTIONS OF PROBLEMS.

NOTE. The following solutions were received too late for publication: Calculus, No. 196, solved by J. Scheffer, and A. H. Holmes; No. 197, solved and discussed by Miss Ida M. Schottenfels. Geometry, No. 255, solved by William Hoover; No. 257, solved by F. R. Honey. Mechanics, No. 179, solved by William Hoover, and J. Scheffer.

ALGEBRA.

234. Proposed by **G. W. GREENWOOD, M. A.** (Oxon). Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

Prove that $(x+n)^n-n(x+n-1)^n+\frac{n(n-1)}{2!}(x+n-2)^n-+\dots\dots=n!$

Solution by the **PROPOSER**.

$\triangle u_x=u_{x+1}-u_x=(E-1)u_x$ where $Eu_x=u_{x+1}$.

Hence $(E-1)x^n=\triangle x^n$, $(E-1)^n x^n=\triangle^n x^n$, *i. e.*, $(x+n)^n-n(x+n-1)^n+\frac{n(n-1)}{2!}(x+n-2)^n-+\dots=\triangle^n x^n!=n!$ by a well known theorem in the Calculus of Finite Differences.*

Also solved by J. Scheffer.

235. Proposed by **WILLIAM HOOVER, Ph. D.**, Athens, Ohio.

Easter Sunday, 1905, was on April 23. How often in the last one hundred years has this occurred, and when?†

Solution by the **PROPOSER**.

The Dominical Letter for April 23, 1905, is A. No other year thus far in the twentieth century had A for its Dominical Letter.

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*Epsteen, MONTHLY, June-July, 1904, p. 132.

†Dr. Zerr adds the remark that this will occur again in 1916 and 2000.